Logarithmically slow expansion of hot bubbles in gases

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We predict a logarithmically slow expansion of hot bubbles in gases in the process of cooling. A model problem is first solved, when the temperature has compact support. Then the temperature profile decaying exponentially at large distances is considered. The periphery of the bubble is shown to remain essentially static (''glassy'') in the process of cooling until it is taken over by a *logarithmically* slowly expanding ''core.'' An analytical solution to the problem is obtained by matched asymptotic expansion. This problem gives an example of how logarithmic corrections enter dynamic scaling.

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The dynamic scaling behavior of extended nonlinear systems out of equilibrium has attracted much attention in different areas of physics [1]. In continuum models dynamic scaling is intimately related to the self-similar asymptotics of nonlinear partial differential equations [2]. Sometimes logarithmic corrections enter dynamic scaling laws [1]. No general scenario for their appearance is known. One can say from experience that they appear in marginal cases, dividing regimes with qualitatively different behavior. The aim of this work is to investigate one particular setting, of general interest, where logarithmic corrections to scaling appear unexpectedly: cooling dynamics of hot bubbles in gases.

Heat transfer in gases, strongly heated locally, looks quite different from the simple picture provided by the linear heat equation. The difference is mainly due to the small-Mach-number conductive cooling flow (CCF) that develops (even at zero gravity) owing to small pressure gradients. The CCF brings in cold gas from the periphery and can strongly modify the cooling dynamics. The importance of CCFs was recognized long ago in astrophysics [3] and in the context of the late stage of strong explosions [4,5]. More recently a quantitative investigation of CCFs began [6–8]. In this paper we predict a new, striking feature of CCFs. If the initial temperature profile rapidly decays at large distances [like $\exp(-k|x|)$, $k>0$], the hot bubble, while cooling down significantly, should expand *logarithmically* slowly.

Starting from the continuity, momentum, and energy equations for an inviscous ideal gas at zero gravity, and employing the small-Mach-number expansion, one arrives at the following nonlinear equation for the scaled gas temperature [6]:

$$\partial_t T = T^2 \partial_x (T^{\nu-1} \partial_x T), \tag{1}$$

where the subscripts t and x stand for partial derivatives (a slab geometry is considered), and ν is the exponent in the assumed power-law temperature dependence of the heat conductivity of the gas [9].

The scaled gas pressure stays constant (and equal to unity) in this approximation, so the scaled gas density is simply $\rho(x,t) = T^{-1}(x,t)$, while the gas velocity is $v(x,t) = T^{\nu}\partial_x T$ [6]. Therefore, once solving Eq. (1) for the temperature, one can easily find all other variables.

Equation (1) has a multitude of similarity solutions:

$$T_\beta(x,t) = t^{(2\beta-1)/(\nu+1)} \theta(x/t^\beta), \tag{2}$$

where β is an arbitrary parameter. Therefore, an interesting selection problem appears, like in many other situations in the nonlinear dynamics of extended systems [2,10].

Equation (1) has appeared in the context of the cooling of the ''fireball'' produced by a strong local explosion in a gas [6–8]. An explosion involves energy release on a time scale short compared to the characteristic acoustic time. In this case the preceding rapid stage of the dynamics produces an inverse power-law dependence of the gas temperature on the distance from the explosion site [4]. It has been shown [6,7] that the exponent of this power law uniquely selects the scaling exponent β. As a result, the fireball expansion exhibits a power law in time.

A different type of local heating occurs when the energy release time is long compared to the acoustic time, but still short compared to the cooling time. In this case the initial temperature profile is more localized, as it reflects the spatial structure of the heating agent (for example, the radial intensity of the laser beam). This new regime will be the focus of this paper. We will see that there is *no* similarity asymptotics to this problem. Instead, the solution approaches a ''quasi-similarity'' asymptotics with logarithmic corrections to scaling.

Consider first a model problem when the initial temperature profile has compact support: $T(x,0) = T_0(x) > 0$ at $x \in [-L,L]$, and zero elsewhere. We will limit ourselves to a temperature-independent heat conductivity $\nu = 0$. Despite this choice, the nonlinearity of Eq. (1) persists. Assume symmetry with respect to $x=0$ and impose the Neumann boundary conditions $\partial_x T(0,t) = \partial_x T(L,t) = 0$ [11]. A local analysis of Eq. (1) near the edge of support of its solution $T(x,t)$ shows that the support remains compact and *unchanged* for

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$t>0$. What is the late-time behavior of the temperature? The constancy of support immediately selects $\beta=0$, so the similarity ansatz becomes $T_0(x,t)=t^{-1}\theta(x)$. Then Eq. (1) yields $\theta(x)=(a^2/2)\cos^2(x/a)$ for $x\in[-L,L], \theta(x)=0$ elsewhere, and $a=\pi L/2$. This simple similarity solution describes the cooling of the hot bubble (and filling it with the dense gas) without *any* change in the bubble size.

Remarkably, $T_0(x,t)$ represents a long-time asymptotics for *any* initial condition that has compact support $[-L,L]$ and obeys the Neumann boundary conditions. We will show here only that this solution is linearly stable with respect to small perturbations, and find the spectrum of the linearized problem. Introduce new variables $u=tT(x,t)$ and $\tau=\ln t$. Equation (1) assumes the form

$$\partial_\tau u=u-(\partial_x u)^2+u\,\partial_{xx}u, \tag{3}$$

while the similarity solution $T_0(x,t)$ becomes a steady-state solution $\theta(x)$. Introducing a small correction $v(x,\tau)$ to this solution and linearizing Eq. (3), we obtain $\partial_\tau v=\hat{L}(\xi)v$, where

$$\hat{L}(\xi)=(1/2)\cos^2\xi\,\partial_{\xi\xi}+\sin 2\xi\,\partial_\xi+2\sin^2\xi \tag{4}$$

and $\xi=x/a$. We look for eigenfunctions in the form of $v(\xi,\tau)=e^{\gamma\tau}\psi_\gamma(\xi)$. The general solution of the resulting ordinary differential equation is

$$\psi_\gamma(\xi)=C_1\cos^2\xi\;{}_2F_1(a_-,a_+,1/2,-\tan^2\xi)+C_2\cos\xi\sin\xi\;{}_2F_1(b_-,b_+,3/2,-\tan^2\xi), \tag{5}$$

where ${}_2F_1$ is the hypergeometric function, C_1 and C_2 are arbitrary constants, $a_\pm=[1\pm(8\gamma+9)^{1/2}]/4$, and $b_\pm=[3\pm(8\gamma+9)^{1/2}]/4$. Requiring that the perturbation remain small compared to the unperturbed solution [and hence vanish like $(\pi/2-\xi)^2$ or faster at $\xi\to\pi/2$], we find the (continuous) spectrum of the linearized problem: $-\infty<\gamma\leqslant-1$. This result proves the linear stability of the similarity solution $T_0(x,t)$. Notice the presence of a gap between the upper edge of the spectrum $\gamma=-1$ and the stability border $\gamma=0$. Going back to physical variables, we find that small *temperature* perturbations around the similarity solution exhibit a power-law decay $t^{\gamma-1}$.

It should be noticed that $\beta=0$ is a marginal case dividing two qualitatively different types of dynamics as described by solutions (2). Indeed, solutions with $\beta>0$ describe power-law *expansions* [6–8], while solutions with $\beta<0$ correspond to power-law *shrinkings* [12]. One can expect logarithmic corrections to appear in the special case $\beta=0$ if the initial condition does *not* have compact support, but decays rapidly enough. Therefore, we assume that the initial temperature profile of the bubble is symmetric with respect to $x=0$ and decays exponentially [13]. We will continue using the new variables and require $u(x,0)\to c\exp(-k|x|)$ at $|x|\to\infty$, where k and c are positive constants. One can always put $k=1$ [14]. We will be interested in a long-time asymptotics of the solution: $\tau\gg1$. Our first important observation is that $u(x,\tau)=c\exp(\tau-x)$ is an *exact* solution of Eq. (3). This traveling-wave solution with a unit speed corresponds to a steady-state solution $T(x)=c\exp(-x)$ in physical variables, and it represents the correct asymptotics of the solution to our problem at $x\to+\infty$. What about the bubble ''core''? We will show that it can be described, at $\tau\gg1$, by a ''quasisimilarity'' solution plus small corrections:

$$u(x,\tau)=u_0(x,\tau)+u_1(x,\tau)+\cdots, \tag{6}$$

where

$$u_0(x,\tau)=\frac{a^2(\tau)}{2}\cos^2\frac{x}{a(\tau)} \tag{7}$$

and $\cdots\ll u_1\ll u_0$. One of our goals is to find an asymptotic expansion for $a(\tau)$.

The leading term of $a(\tau)$ can be guessed immediately. Indeed, expansion of $u_0(x,\tau)$ in powers of $x-\pi a(\tau)/2$ near the point $x=\pi a(\tau)/2$ begins with the term $(1/2)[x-\pi a(\tau)/2]^2$. This is a wave traveling with speed $\pi\dot{a}/2$ along the x axis. Therefore, it is natural to look for a *general* traveling-wave solution $v(x,\tau)=V(x-\tau)$ of Eq. (3) with a unit speed and require that it behave like $(z+\text{const})^2/2$ at $z\to-\infty$ and like $c\exp(-z)$ at $z\to+\infty$, where $z=x-\tau$. If such a solution exists, we can match it with the leading term of the quasisimilarity solution (7) in the region $1\ll-(x-\pi a/2)\ll a, 1\ll-z$, once

$$a(\tau)=2\tau/\pi=(2/\pi)\ln t. \tag{8}$$

Equations (7) and (8) have important implications. First, the temperature scaling at the bubble center acquires a logarithmic correction. Second, the bubble core expands logarithmically slowly. We will show in the following that these are indeed correct results, calculate the subleading and subsubleading terms for $a(\tau)$, and find other attributes of the asymptotic solution.

The general traveling-wave solution of Eq. (3), $V(z)$, obeys the second-order equation

$$-V_z=V-V_z^2+VV_{zz}, \tag{9}$$

which is soluble analytically. One integration yields

$$V^{-1}(dV/dz)=-1-W[-\exp(-1-V^{-1})], \tag{10}$$

where $W(\eta)$ is the product log function defined as the solution of equation $We^W=\eta$ (see, e.g., Ref. [15], p. 751). The arbitrary constant in Eq. (10) has been chosen to satisfy the required asymptotic behavior $V(z)\to(z+\text{const})^2/2$ at $z\to-\infty$. Notice that, as $V>0$ and $V_z<0$, we should work with the negative branch of the product log function: $\eta<0$ and $W(\eta)<0$.

Integrating Eq. (10), we obtain the traveling-wave solution in an implicit form:

$$J(V)+z+C=0, \tag{11}$$

where

$$J(V)=\int_1^{U(V)}\frac{d\zeta}{1-\zeta-e^{-\zeta}}, \tag{12}$$

$U(V)=-\ln\{-W[-\exp(-1-V^{-1})]\}$ and C is an arbitrary constant.

To understand the asymptotic behavior of this solution at $z\to-\infty$ and $z\to+\infty$, we need to know the asymptotics of $J(V)$. After some algebra we obtain

$$J(V)=\begin{cases}\ln V+\Delta_1+O(V),\\ (2V)^{1/2}+\dfrac{1}{3}\ln V+\Delta_2+O(V^{-1/2}),\end{cases}\qquad(13)$$

at $V\to+0$ and $V\to+\infty$, respectively. Here

$$\Delta_1=\int_1^\infty\frac{(1-e^{-\zeta})d\zeta}{\zeta(1-\zeta-e^{-\zeta})}=-1.460\,744\,00\ldots,$$

$\Delta_2=-2-(1/3)\ln(2e)-\Delta_3$, and

$$\Delta_3=\int_0^1\left(\frac{1}{1-\zeta-e^{-\zeta}}+\frac{2}{\zeta^2}+\frac{2}{3\zeta}\right)d\zeta=-0.053\,618\,92\ldots.$$

Using Eqs. (11) and (13), we obtain

$$V(z)=e^{-z-C-\Delta_1}+O(e^{-2z})\quad\text{at}\quad z\to+\infty\qquad(14)$$

and

$$V(z)=(1/2)(z+C+\Delta_2)^2+(2/3)(z+C+\Delta_2)\ln[(|z+C+\Delta_2|)/\sqrt{2}]+O(\ln^2|z+C+\Delta_2|)\quad\text{at}\quad z\to-\infty.\qquad(15)$$

The required asymptotic behavior $V\to c\exp(-z)$ at $z\to+\infty$ selects $C=-\Delta_1-\ln c$, so the traveling-wave solution (11) is now fully determined. After some rearrangement, we rewrite the asymptotics (14) and (15) as

$$V(z)=ce^{-z}+O(e^{-2z})\quad\text{at}\quad z\to+\infty,\qquad(16)$$

and

$$V(z)=(1/2)(z-\Delta)^2+(2/3)(z-\Delta)\ln|z-\Delta|+O(\ln^2|z|)\quad\text{at}\quad z\to-\infty,\qquad(17)$$

where $\Delta=\Delta_1-\Delta_2+(1/3)\ln 2+\ln c$.

The leading term in Eq. (16) corresponds to a steady-state solution in the physical variables, while the subleading term is *exponentially* small with respect to the leading one. This essentially static (''glassy'') behavior of the solution at large distances reflects *effective* diffusion choking at small temperatures.

Now let us return to the bubble core description, Eq. (7). Our basic assumption here (supported by the results) is that, in the asymptotic stage $\tau\gg1$, the terms $u_0,u_1,\ldots$ depend on time only through the time dependences of a, of a and $\dot a$, of $a,\dot a,\ldots$, respectively. The small parameter of this expansion is $\dot a/a$. In the zeroth approximation of this perturbation scheme, u_0 obeys Eq. (3) without the time derivative term. In the first approximation we obtain the following linear equation:

$$\hat L u_1(\xi)=a\dot a\cos^2\xi(1+\xi\tan\xi),\qquad(18)$$

where we have again used $\xi=x/a$.

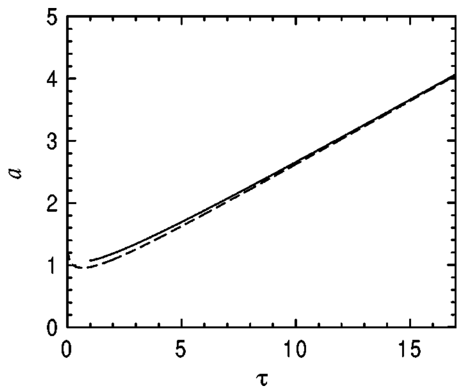

FIG. 1. The amplitude a versus the new time $\tau=\ln t$ found numerically (dashed line) and analytically (solid line). The parameters are described in the text.

The zero modes of the operator $\hat L$ are $\Upsilon(\xi)=\cos^2\xi+\xi\cos\xi\sin\xi$ and $\Phi(\xi)=\sin\xi\cos\xi$. Looking for a general solution of Eq. (18) in the form of $u_1=C_1(\xi)\Upsilon(\xi)+C_2(\xi)\Phi(\xi)$ and defining $a(\tau)$ by the condition $u(0,\tau)=a^2(\tau)/2$, we arrive at

$$u_1(\xi)=-(2/3)a\dot a\cos^2\xi\{(\xi\tan\xi-1)\ln\cos\xi+2\ln(2/e)\xi\tan\xi+\tan\xi\,\mathrm{Im}[\mathrm{Li}_2(-e^{2i\xi})]\},\qquad(19)$$

where $\mathrm{Li}_2(x)=\Sigma_{k=1}^\infty k^{-2}x^k$ is the dilogarithm (see, e.g., Ref. [15], p. 743). In the vicinity of $\xi=\pi/2$,

$$u_1=\frac{\pi}{3}a\dot a\tilde\xi\ln\frac{4|\tilde\xi|}{e^2}+a\dot aO(|\tilde\xi|^3\ln|\tilde\xi|),\qquad(20)$$

where $\tilde\xi=\xi-\pi/2$. Too close to $\xi=\pi/2$ the correction u_1 and its derivatives become larger than the zero-order solution u_0 and its corresponding derivatives, so the perturbation procedure breaks down. Therefore, the bubble core solution [Eqs. (6), (7), and (19)] should be matched with the traveling-wave solution [Eq. (11)] in the region where $|x-\pi a/2|$ is small enough (so that the leading term of the asymptotics of u_0 is much larger than the subleading terms) but, on the other hand, large enough (so that u_1 is small compared to u_0). Working in this region and collecting the leading contributions from u_0 and u_1, we obtain, after some rearrangement,

$$u=\frac{1}{2}\tilde x^2+\frac{\pi}{3}\dot a\tilde x\ln|\tilde x|+\frac{1}{a^2}O(\tilde x^4)+\frac{\dot a}{a^2}O(|\tilde x|^3\ln|\tilde x|)+O(\dot a^2\ln^2 a)O(\ln|\tilde x|)+\cdots,\qquad(21)$$

where $\tilde x=x-(\pi/2)a-(\pi/3)\dot a\ln(e^2a/4)$.

Now we can perform the matching procedure. We require that the $z\to-\infty$ asymptotics of the traveling wave solution, Eq. (17), coincide with the asymptotics (21) of the bubble core solution. This yields

$$a(\tau)=\frac{2}{\pi k}\left[\tau-\frac{2}{3}\ln\frac{\tau}{4\pi}+B+\ln ck^2+o(1)\right],\qquad(22)$$

where $B=1+\Delta_1+\Delta_3=-0.514\,362\,926\ldots, o(1)$ denotes terms that vanish as $\tau\to\infty$ and we have restored the k dependence [14]. We see that the leading term in $a(\tau)$ is logarithmic in physical time t, and it coincides with Eq. (8). The subleading term behaves like $\ln\tau\sim\ln\ln t$, while the sub-subleading term is constant.

The matching region is determined by the requirements that the subleading term in Eq. (21) be much less than the leading term, but much greater than the rest of terms. These yield an approximate condition $\ln^2 a \ll -x+(\pi/2)a \ll a^{2/3}$, so that the matching region expands as $\tau\to\infty$.

We checked the asymptotic solution numerically by directly solving Eq. (3) in the new variables. This enabled us to reach $\tau\sim 20$, that is $t\sim 5\times 10^8$. Equation (3) was solved on the interval $x\in(0,9)$ subject to the Neumann boundary conditions. The initial condition was $u(x,\tau=0)=2\exp[-3(x^2+0.1)^{1/2}]$, so that $c=2$ and $k=3$. The system length was large enough for the solution to enter the asymptotic regime before the expanding ''core'' reaches the boundary $x=9$. Figure 1 compares $a=[2u(0,\tau)]^{1/2}$ found numerically with the prediction of Eq. (22). At long times the agreement is excellent. We also verified other attributes of the asymptotic solution.

In separate simulations [16], the evolution of the same initial condition was investigated in the framework of the complete set of gas dynamic equations. In the small-Mach-number regime the results essentially coincide (except for an acoustic transient) with those obtained with Eq. (1).

In conclusion, we predict a logarithmically slow expansion of hot bubbles in gases. By constructing an asymptotic solution that matches a ''quasisimilarity'' inner solution and a ''glassy'' outer solution, we have been able to see how logarithmic corrections enter dynamic scaling. We hope that this study will motivate experimental work on hot bubble dynamics.

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[12] Shrinking is possible when a cold ''cloud'' is heated by a hot and underdense peripheral gas. A problem mathematically equivalent to this one was considered in Ref. [17].

[13] In reality, the temperature should approach a *finite* value at large distances. Correspondingly, the solution we are interested in represents an *intermediate* asymptotics [2].

[14] If $u(x,\tau)$ is a solution of Eq. (3), then $\lambda^2 u(x/\lambda,\tau)$ is also a solution for any $\lambda>0$. This property enables one to restore the k dependence in the final results.

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